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ANALYSIS OF THE THREE-DIMENSIONAL STATES OF STRESS AND STRAIN OF CIRCULAR CYLINDRICAL SHELLS. CONSTRUCTION OF REFINED APPLIED THEORIES

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The nonaxisymmetric problem of elasticity theory for circular cylindrical shells loaded along the endface surface Γ_2 is considered. By using the method of trigonometric series expansions, homogeneous solutions of closed $(\Gamma_2 : z = \pm l)$ and open $(\Gamma_2 : \varphi = + \varphi_0)$ shells are studied as their thickness decreases.

It is proved that the state of stress of a closed shell includes four parts: (1) an elementary state of stress penetrating into the shell without attenuation, (2) a slowly attenuated principal state of stress, (3) a rapidly attenuating state of stress (edge effect of shells), (4) a boundary layer type of state of stress.

In the case of an open cylindrical shell subjected to a periodic loading with period l_0, there are states of stress of types (1), (3) and (4). The rate of attenuation of the edge effects hence depend essentially on the number of the term of the trigonometric series as well as on the quantity l_0. In both cases asymptotic expansions are presented of the components of the states of stress and strain.

On the basis of the exact solution of the three-dimensional problem, a refined applied theory is given for a circular cylindrical shell, which is intended to reduce the stress from the endface surface Γ_2. Applied theories reducing the stresses from the cylindrical portions of the shell boundary were considered earlier in [1].

1. Construction of homogeneous solutions. Let us consider the arbitrary strain of an elastic isotropic shell bounded by coaxial circular cylinders Γ_1 of radii R_1 and R_2 $(R_1 < R_2)$ and an endface surface Γ_2. Let us assume that the stress resultants applied to the boundary Γ_2 form a system statically equivalent to zero, and the boundary Γ_1 is stress-free. As the initial relationships let us take expressions for the displacements and stresses obtained in [1] on the basis of Lur'e's symbolic writing [2]

$$u = R_3\{Z_\nu' A - \xi^{-1} Z_\nu \partial_2 N - \xi Z_\nu C + *\}$$
$$v = R_3\{\xi^{-1} Z_\nu \partial_2 A + Z_\nu' N + *\} \qquad (1.1)$$
$$w = R_3\{Z_\nu A + [\xi Z_\nu' + 2\varkappa Z_\nu] C + *\}$$

$$\sigma_r = 2Gp\{[Z_\nu(\nu^2\xi^{-2} - 1) - \xi^{-1} Z_\nu'] A + (\xi^{-2} Z_\nu - \xi^{-1} Z_\nu')\partial_2 N + [(1-\varkappa) Z_\nu - \xi Z_\nu'] C + *\}$$
$$\tau_{r\varphi} = Gp\{2(-\xi^{-2} Z_\nu + \xi^{-1} Z_\nu')\partial_2 A + [Z_\nu(2\nu^2\xi^{-2} - 1) - 2\xi^{-1} Z_\nu'] N - Z_\nu \partial_2 C + *\} \qquad (1.2)$$
$$\tau_{rz} = Gp\{2Z_\nu' A - \xi^{-1} Z_\nu \partial_2 N + [Z_\nu(\nu^2\xi^{-1} - 2\xi) + 2\varkappa Z_\nu'] C + *\}$$

$$\sigma_z = 2Gp\{Z_\nu A + [(2+\varkappa) Z_\nu + \xi Z_\nu'] C + *\}$$
$$\tau_{z\varphi} = Gp\{2\xi^{-1} Z_\nu \partial_2 A + Z_\nu' N + (2\xi^{-1}\varkappa Z_\nu + Z_\nu')\partial_2 C + *\} \qquad (1.3)$$
$$\sigma_\varphi = 2Gp\{(-\nu^2\xi^{-2} Z_\nu + \xi^{-1} Z_\nu') A + (-\xi^{-2} Z_\nu + \xi^{-1} Z_\nu')\partial_2 N + (1-\varkappa) Z_\nu C + *\}$$
$$\left(p = \frac{\partial}{\partial\zeta},\ \partial_2 = \frac{\partial}{\partial\varphi},\ \nu = -i\partial_2,\ \xi = p\rho,\ \zeta = \frac{z}{R_3},\ \rho = \frac{r}{R_3},\ \varkappa = \frac{2(m-1)}{m}\right)$$

Here m is the Poisson number, A, N, C are arbitrary functions of ζ and φ, $Z_\nu(\xi)$ the cylindrical operator function (see e.g. [1]), R_3 the characteristic dimension, and the asterisk denotes the analogous expressions obtained by replacing $Z_\nu(\xi), A, N, C$ by the functions $X_\nu(\xi)$, A^*, N^*, C^*.

Let us extract the class of homogeneous solutions out of (1.1)–(1.3), i.e. solutions for which there are no stresses on the boundary Γ_1

$$\sigma_r = 0, \quad \tau_{r\varphi} = 0, \quad \tau_{rz} = 0 \quad \text{when } r = R_1, R_2 \qquad (1.4)$$

Substituting $\sigma_r, \tau_{r\varphi}, \tau_{rz}$ from (1.2) into (1.4), we obtain a system of homogeneous differential equations of infinitely high order in the unknown functions $A, N, \ldots C^*$

$$d_{11}A + d_{12}N + \ldots + d_{16}C^* = 0$$
$$\cdots\cdots\cdots \qquad (1.5)$$
$$d_{61}A + d_{62}N + \ldots + d_{66}C^* = 0$$

where the d_{ik} denote appropriate operators. Following [3–5], it is possible to take as the solution of (1.5)

$$A = A_{1k}\Psi(\zeta,\varphi), \quad N = A_{2k}\Psi(\zeta,\varphi), \ldots, \quad C^* = A_{6k}\Psi(\zeta,\varphi) \quad (1.6)$$

The operators A_{ik} here are cofactors of elements of the kth rows of the operator-determinant Q of the system (1.5); $\Psi(\zeta,\varphi)$ is a stress function satisfying the equation

$$Q\,\Psi(\zeta,\varphi) = 0 \qquad (1.7)$$

Equation (1.7) determines a countable set of solutions $\Psi_k(\zeta,\varphi)$, and their corresponding functions $A_k, N_k, \ldots, C_k^*$ form homogeneous solutions for a circular cylindrical shell after substitution into (1.1)–(1.3).

Evaluating the determinant of the system (1.5), we obtain

$$Q = p^5 \sum_{k,j=0}^{1} \sum_{i,\mu=0}^{1} L_{kj}(L_{i\mu}^2 P_{kj,i\mu} + P_{kj}) \qquad (1.8)$$

$$L_{ki} = [J_\nu^{(k)}(\xi_1) K_\nu^{(i)}(\xi_2) - J_\nu^{(i)}(\xi_2) K_\nu^{(k)}(\xi_1)]\, p$$
$$f^{(0)}(x) = f(x),\ f^{(1)}(x) = df(x)/dx,\ \xi_1 = pR_1/R_3,\ \xi_2 = pR_2/R_3 \quad (1.9)$$

Here J_ν is a Bessel function of the first kind, K_ν a Weber-Schlaeffli function of the second kind, $P_{kj,i\mu}$ and P_{kj} are expressions of the following kind:

$$\sum_{k=0}^{6}\sum_{i=0}^{6} a_{ki}\frac{\gamma^{2k}}{p^i} \qquad \left(\gamma = \frac{\nu}{p},\ 2k+i \leqslant 12\right) \tag{1.10}$$

Now putting $R_3 = \sqrt{R_1 R_2}$ and taking into account that the operator Q is an entire function in this case of not only D^2, p^2 but also ε ($\varepsilon = 0.5 \ln (R_2 / R_1)$, $D^2 = p^2 + \partial_2^2$), we represent the right side of the relationship (1.8) as a power series in ε^2. Calculations yield

$$Q = \varepsilon^3 \sum_{k=0}^{\infty} \varepsilon^{2k}\Omega_k (D^2, p^2) \tag{1.11}$$

$$\Omega_0 = 2b_0 p^4,\ \Omega_1 = b_0 (8 - 4D^2)\, p^4 + {}^{16}/_3 (D^2 + 1)\, p^4 - {}^4/_3 (D^4 + D^2 + p^2)^2$$
$$\Omega_2 = {}^8/_5 D^{10} + {}^{112}/_{45} D^8 - ({}^{32}/_{45}\varkappa + {}^{16}/_{45}) D^6 p^2 + ({}^{52}/_{15} b_0 - {}^{32}/_5)\, D^4 p^4 + \Omega_2^*$$
$$\Omega_3 = -\,{}^{808}/_{945} D^{12} + \Omega_3^* \quad (2b_0 = \varkappa^2 - 4\varkappa) \tag{1.12}$$

Here Ω_2^* and Ω_3^* are lower order operators than those written down.

If expansions of the quantities L_{jk} in powers of p are utilized

$$L_{00} = pb_0^* - {}^1/_4 p^3 b_1 + {}^1/_{16} p^5 (b_1 \operatorname{ch} 2\varepsilon - {}^1/_2 b_2) + \ldots \tag{1.13}$$
$$L_{11} = -\, b_0^* \nu^2 / p + {}^1/_2 p\, ({}^1/_2 b_1 - a_0 \operatorname{sh} 2\varepsilon - b_0^* \operatorname{ch} 2\varepsilon) +$$
$$+ {}^1/_8 p^3 (b_0^* \operatorname{sh}^2 2\varepsilon - {}^3/_2\, b_1 \operatorname{ch} 2\varepsilon + b_2) + \ldots$$
$$L_{01} = e^{-\varepsilon}\{ a_0 - {}^1/_2 p^2 (b_0^* \operatorname{sh} 2\varepsilon + {}^1/_2\, b_1) + {}^1/_{16}\, p^4 (b_1 \operatorname{ch} 2\varepsilon + e^{\varepsilon} b_1 - b_2) + \ldots\}$$
$$L_{10} = e^{\varepsilon}\{ -a_0 + {}^1/_2 p^2 (b_0^* \operatorname{sh} 2\varepsilon - {}^1/_2 b_1) + {}^1/_{16} p^4 (b_1 \operatorname{sh} 2\varepsilon + e^{-\varepsilon} b_1 - b_2) + \ldots\}$$
$$(a_0 = \operatorname{ch} 2\varepsilon\nu, \qquad b_0^* = \operatorname{sh} 2\varepsilon\nu / \nu$$
$$b_k = [\operatorname{sh} 2\varepsilon\, (k + \nu) / (k + \nu) - \operatorname{sh} 2\varepsilon\, (k - \nu) / (k - \nu)]\ {}^1 / \nu$$

then we obtain by substituting (1.13) into (1.8)

$$Q\ (\varepsilon, \partial_2^2, p^2) = {}^1/_8\, (\partial_2 + \partial_2^3) \sin 2\partial_2\varepsilon\, (\sin^2 2\partial_2\varepsilon - \partial_2^2 \operatorname{sh}^2 2\varepsilon) +$$
$$+ p^2\partial^2 (1 + \partial_2^2)\, U_1 (\varepsilon, \partial_2^2) + p^4 U_2 (\varepsilon, \partial_2^2) + \ldots \tag{1.14}$$

where the entire functions U_1 $(\varepsilon, \partial_2^2)$ and U_2 $(\varepsilon, \partial_2^2)$ do not vanish for $\partial_2^2 = -1$ and $\partial_2^2 = 0$.

By virtue of (1.10) the operator Q can be written as

$$Q = p^5\{ Q^* L_{00} \Delta_1^2 \Delta_2^2\} + p^4\{2 Q^* [L_{10} e^{\varepsilon}\} (1 + \varkappa\gamma^2 e^{2\varepsilon})\, \Delta_2^2 +$$
$$+ L_{01} e^{-\varepsilon} (1 + \varkappa\gamma^2 e^{-2\varepsilon})\ \Delta_1^2] + 8\varkappa\{\operatorname{sh}^2 2\varepsilon\, (L_{10} e^{\varepsilon} \Delta_2 + L_{01} e^{-\varepsilon} \Delta_1)\, \gamma^2\} +$$
$$+ p^3\{2 Q^* [2L_{11} (1 + 2\varkappa\gamma^2 \operatorname{ch} 2\varepsilon + \varkappa^2\gamma^4) - L_{00}\gamma^2 (\Delta_1\Delta_2 + 2 \operatorname{sh}^2 2\varepsilon)] -$$
$$- \varkappa L_{00} \langle (\gamma^4 \operatorname{ch} 2\varepsilon - 2\gamma^2 + \operatorname{ch} 2\varepsilon)\, (2L_{11}^2 - 2L_{00}^2\gamma^4 + L_{01}^2 + L_{10}^2 - 2 \operatorname{ch} 2\varepsilon) +$$
$$+ 2\gamma^2 L_{00}^2 (\Delta_1\Delta_2 + 2\operatorname{sh}^2 2\varepsilon) + (\gamma^4 - 1) \operatorname{sh} 2\varepsilon\ [L_{01}^2 - L_{10}^2 - 2\gamma^2\rangle \times$$
$$\times \langle (L_{01}^2 e^{2\varepsilon} - L_{10}^2 e^{-2\varepsilon})] \rangle - 2\varkappa \operatorname{sh}^2 2\varepsilon\ [L_{00} - 8L_{11}\gamma^2 - (9 + 2\varkappa) L_{00}\gamma^4]\} + \ldots \tag{1.15}$$

where

$$Q^* = L_{00}^2 \Delta_1\Delta_2 + L_{10}^2 \Delta_2 + L_{01}^2 \Delta_1 + L_{11}^{2\varepsilon} - e^{-2\varepsilon}\Delta_1 - e^{2\varepsilon}\, \Delta_2$$
$$\Delta_1 = 1 - \gamma^2 e^{2\varepsilon}, \qquad \Delta_2 = 1 - \gamma^2 e^{-2\varepsilon}$$

Finally, the operator Q admits of yet another representation which is specified by the appropriate expansion of the quantities L_{jk}

$$Q = (2\varepsilon)^{-3} / 8 \sum_{k=0}^{\infty} (2\varepsilon)^{2k} Q_k (D_*^2, p_*^2) \tag{1.16}$$

where

$$Q_0 = D_*^3 \sin D_* (\sin^2 D_* - D_*^2) \qquad (D_*^2 = (2\varepsilon)^2 D^2, \ p_* = 2\varepsilon p)$$

$$\begin{aligned} Q_1 = {} & \sin^3 D_* \{p_*^4 [(-{}^{41}/_8 + 4\varkappa + \varkappa^2) D_*^{-3} - {}^3/_4 D_*^{-1}] + \\ & + p_*^2 [({}^1/_2 - 6\varkappa) D_*^{-1} + {}^3/_4 D_*] + D_*\} + \\ & + \sin^2 D_* \cos D_* \{p_*^4 [(-{}^9/_8 - 3\varkappa) D_*^{-2} - {}^1/_8] + \\ & + p_*^2 [(-{}^1/_2 + 2\varkappa) + {}^1/_4 D_*^2]\} + \sin D_* \{p_*^4 [({}^7/_8 - 3\varkappa) D_*^{-1} + {}^3/_2 D_*] + \\ & + p_*^2 [(-{}^3/_2 + 2\varkappa) D_* - {}^5/_4 D_*^3] - D_*^3 - {}^1/_3 D_*^5\} + \\ & + \cos D_* \{p_*^4 [({}^{43}/_8 - 2\varkappa) + {}^1/_{24} D_*^2] + p_*^2 [(-{}^7/_2 + 2\varkappa) D_*^2 - {}^1/_{12} D_*^4]\} \end{aligned} \tag{1.17}$$

Let us study the singularities in the behavior of the homogeneous solutions by using the method of trigonometric series expansions. In the case of a finite hollow cylinder $(\Gamma_2 : z = \pm l)$ the stress function $\Psi_1 (\zeta, \varphi)$ can be sought by putting

$$\Psi_1 (\zeta, \varphi) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} H_{ki} \begin{Bmatrix} \operatorname{sh} \lambda_{ki} \zeta \sin k\varphi \\ \operatorname{ch} \lambda_{ki} \zeta \cos k\varphi \end{Bmatrix} \tag{1.18}$$

For a cylindrical panel $(\Gamma_2 : \varphi = \pm \varphi_0)$ subjected to a periodic loading with period l_0, we will seek the stress function $\Psi_2 (\zeta, \varphi)$ in the form

$$\Psi_2 (\zeta, \varphi) = \sum_{i=0}^{\infty} \sum_{m=0}^{\infty} H_{ni}^* \begin{Bmatrix} \operatorname{sh} k_{ni} \varphi \sin n_m \zeta \\ \operatorname{ch} k_{ni} \varphi \cos n_m \zeta \end{Bmatrix} \qquad \left(n_m = m \frac{\pi}{l_0}\right) \tag{1.19}$$

The H_{ki} and H_{ni}^* in (1.18), (1.19) are arbitrary constants, and the λ_{ki} and k_{ni} some parameters.

Substituting (1.18) and (1.19) into (1.7), we obtain characteristic equations in λ_{ki} and k_{ni}:

$$Q (\varepsilon, -k^2, \lambda_{ki}^2) = 0, \qquad Q (\varepsilon, k_{ni}^2, -n_m^2) = 0 \tag{1.20}$$

2. Analysis of the roots of the characteristic equation of a closed cylindrical shell. Let us analyze the roots of the first transcendental equation in (1.20) as $\varepsilon \to 0$. Let us first examine the case of small k $(k < \varepsilon^{-1/2})$. Let us seek the roots which have a finite limit as $\varepsilon \to 0$. If such roots exist, then their limit values λ_{ki0} are evidently found from the limit equation

$$[\varepsilon^{-3} Q (\varepsilon, -k^2, \lambda_{ki0}^2)] \,|_{\varepsilon=0} = 0$$

which has the form $2b_0 \lambda_{ki0}^4 = 0$ in the case under consideration. We hence conclude that (1.20) determines four vanishingly small roots for every k as $\varepsilon \to 0$. Utilizing this property for small λ_{ki} and ε, we write the first equation in (1.20) as

$$\begin{aligned} \varepsilon^3 \{ 2b_0 \lambda_{ki}^4 + \varepsilon^2 [- {}^4/_3 k^4 (k^2 - 1)^2 + {}^{16}/_3 \lambda_{ki}^2 k^2 (k^2 - 1)^2 + 4 b_0 \lambda_{ki}^4 (k^2 + 2) - \\ - 8 \lambda_{ki}^4 k^2 (k^2 - 1) + \ldots] + \varepsilon^4 [- {}^8/_{45} k^4 (k^2 - 1)^2 (4 + 9k^2) + \ldots] + \ldots\} = 0 \end{aligned} \tag{2.1}$$

From (2.1) the following asymptotic expansion results

$$\begin{aligned} \lambda_{ki} = \varepsilon^{1/2} p_0, \quad p_0 = \lambda_{ki0} + \varepsilon \lambda_{ki1} + \varepsilon^2 \lambda_{ki2} + \ldots, \\ \lambda_{ki0}^4 - {}^2/_3 k^4 (k^2 - 1)^2 b_0^{-1} = 0 \\ \lambda_{ki1} = - {}^2/_3 \lambda_{ki0}^{-1} k^2 (k^2 - 1)^2 b_0^{-1} \end{aligned} \tag{2.2}$$

$$\lambda_{ki2} = \lambda_{ki0}\,[{}^1/_3 b_0^{-1}(k^2-1)(4k^2-1) - {}^1/_5 k^2 - {}^{13}/_{15}] \qquad \text{(cont.)}$$

Taking account of (2.2) and (1.14), we easily see that $\lambda_{0i} = 0$ and $\lambda_{1i} = 0$ are exact quadruple roots.

Now, let us assume that all the remaining roots $\lambda_{ki} \to \infty$ as $\varepsilon \to 0$. Then, by using the asymptotic expansions of cylindrical functions of an argument tending to infinity and with a small index, (1.20) can be given the form

$$\begin{gathered}[\alpha_{ki}^3 \sin(a\alpha_{ki})\,(\sin^2(a\alpha_{ki}) - (a\alpha_{ki})^2)] + \varepsilon^2[\alpha_{ki}\, \sin^3(a\alpha_{ki})({}^{11}/_2 - 8\varkappa + \\ + 4\varkappa^2 - 6k^2) - \alpha_{ki}^2 \sin^2(a\alpha_{ki})\cos(a\alpha_{ki})({}^{13}/_2 + 4\varkappa + 6k^2) - \qquad (2.3)\\ - \alpha_{ki}^3 \sin(a\alpha_{ki})({}^{13}/_2 + 4\varkappa - 10k^2) + \alpha_{ki}^4\cos(a\alpha_{ki})({}^{15}/_2 + 2k^2)] + \ldots = 0\\ (\alpha_{ki} = 2\lambda_{ki}\varepsilon,\ a = \varepsilon^{-1}\,\mathrm{sh}\,\varepsilon)\end{gathered}$$

The limit relationships $\alpha_{ki} \to 0$, $d_{ki} \to \mathrm{const}$ and $\alpha_{ki} \to \infty$ are possible for the quantity α_{ki} for $\varepsilon \to 0$ and $\lambda_{ki} \to \infty$.

In the first case $\alpha_{ki} \to 0$ for $\varepsilon \to 0$. Taking account of this property for small α_{ki} and ε, we write (2.3) as

$$\begin{gathered}[-{}^1/_3\alpha_{ki}^8 + {}^1/_{10}\alpha_{ki}^{10} - {}^{101}/_{7560}\alpha_{ki}^{12} + \ldots] + \qquad (2.4)\\ + \varepsilon^2\,[8b_0\alpha_{ki}^4 + ({}^{16}/_3 k^2 - 4b_0)\,\alpha_{ki}^6 + ({}^{13}/_{15} b_0 - {}^{16}/_{15} - {}^8/_{45}\varkappa - 2k^2)\,\alpha_{ki}^8 + \ldots] + \\ + \varepsilon^4\,[32\alpha_{ki}^4(b_0 + {}^1/_2 k^2 b_0 + k^2 - k^4) + \ldots] + \ldots = 0\end{gathered}$$

The following asymptotic expansion results from (2.4)

$$\begin{gathered}\lambda_{ki} = \varepsilon^{-1/2} p_1, \quad p_1 = \alpha_{ki0} + \varepsilon\alpha_{ki1} + \varepsilon^2\alpha_{ki2} + \ldots, \qquad \alpha_{ki0}^4 - {}^3/_2 b_0 = 0 \qquad (2.5)\\ \alpha_{ki1} = (k^2 - {}^3/_{10} b_0)\,\alpha_{ki0}^{-1},\\ \alpha_{ki2} = [(k^2 - {}^2/_3 k^4) b_0^{-1} + {}^{167}/_{2100} b_0 - {}^2/_{15}\varkappa + {}^1/_5]\alpha_{ki0}\end{gathered}$$

Let us examine the second case $\alpha_{ki} \to \alpha_{ki}^*$ as $\varepsilon \to 0$. Then, as is easy to see from (2.3), the α_{ki}^* satisfies the equation

$$(a\alpha_{ki}^*)^{-5}\sin(a\alpha_{ki}^*)\,(\sin^2(a\alpha_{ki}^*) - (a\alpha_{ki}^*)^2) = 0 \qquad (2.6)$$

It should be noted that (2.6) agrees with the equation governing the index of the boundary-layer edge effects in slab theory [6]. Equation (2.6) has a countable set of roots, hence (2.3) also has a countable set of roots such that $\lambda_{ki}\varepsilon \to \mathrm{const}$ as $\varepsilon \to 0$. Refined values values of the mentioned roots can be obtained by using the expansion

$$\lambda_{ki} = p_2\,(2\,\mathrm{sh}\,\varepsilon)^{-1}, \quad p_2 = x_1 + \varepsilon^2\delta_{k2} + \varepsilon^4\delta_{k4} + \ldots \qquad (2.7)$$

$$\delta_{k2} = (4k^2 + 4\varkappa - 1)(2x_1)^{-1} - 8b_0\,(\sin 2x_1 - 2x_1)^{-1}\,(\sin^2 x_1 - x_1^2)\,/\,x_1^4 = 0$$

$$\begin{gathered}p_2 = x_0 + \varepsilon^2\delta_{k2}^* + \varepsilon^4\delta_{k4}^* + \ldots \qquad (2.8)\\ \delta_{k2}^* = (4k^2 + 15)\,(2x_0)^{-1}, \quad \sin x_0\,/\,x_0 = 0\end{gathered}$$

Let us show that the third case is not realizable. Indeed, it is seen from (2.6) that if $\varepsilon \to 0$, then compliance with the asymptotic equality $\sin(a\alpha_{ki})\,(\sin^2(a\alpha_{ki}) - (a\alpha_{ki})^2) \sim 0$ is impossible for α_{ki} tending continuously to infinity.

Turning to the case of average k ($\varepsilon^{-1/2} \lesssim k < \varepsilon^{-1}$), let us introduce the quantities $\lambda = \lambda_{ki}\sqrt{\varepsilon}$ and $k_0 = k\sqrt{\varepsilon}$. The first characteristic equation (1.20) becomes in the new notation

$$(2b_0\lambda^4 - {}^4/_3\Delta^8) + \varepsilon\,[-{}^8/_3\Delta^6 - {}^8/_3\Delta^4\lambda^2 + ({}^{16}/_3 - 4b_0)\,\Delta^2\lambda^4 + {}^8/_5\Delta^{10}] +$$
$$+ \varepsilon^2\,[(8b_0 + 4)\,\lambda^4 - {}^4/_3\Delta^4 - {}^8/_3\Delta^2\lambda^2 + {}^{112}/_{45}\Delta^8 - ({}^{16}/_{45} + {}^{32}/_{45}\varkappa)\,\Delta^6\lambda^2 -$$
$$- ({}^{32}/_5 - {}^{52}/_{15}b_0)\,\Delta^4\lambda^4 - {}^{808}/_{945}\Delta^{12}] + \ldots = 0 \qquad (2.9)$$
$$(\Delta^2 = \lambda^2 - k_0^2)$$

Seeking λ as a power series in ε we obtain

$$(2.10)$$
$$\lambda_{ki} = \varepsilon^{-1/2}(\lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \ldots), \quad \lambda_0^2 - \varkappa_k\lambda_0 - k_0^2 = 0,\ \varkappa_k^4 - {}^3/_2 b_0 = 0$$
$$\lambda_1 = \{\lambda_0\varkappa_k^2\,({}^1/_5 - {}^1/_3 b_0^{-1}) + {}^1/_2\varkappa_k^{-1} + \varkappa_k k_0^2\,({}^1/_5 - {}^2/_3 b_0^{-1})\}\,(1 - 2\lambda_0\varkappa_k^{-1})^{-1}$$
$$\lambda_2 = \{\lambda_0[\,-{}^{167}/_{2100}b_0 + {}^2/_{15}\varkappa - {}^1/_5 - {}^7/_{12}b_0^{-1} + \varkappa_k^2 k_0^2\,(-{}^{709}/_{3150} + {}^{16}/_{45}b_0^{-1}\varkappa -$$
$$- {}^2/_3 b_0^{-1} - {}^4/_9 b_0^{-2}) + k_0^4\,(-{}^{82}/_{1575} - {}^8/_{15}b_0^{-1} + {}^8/_9 b_0^{-2})] + \varkappa_k\,[{}^2/_3 b_0^{-1} +$$
$$+ \varkappa_k^2 k_0^2\,(-{}^{167}/_{3150} + {}^4/_{45}b_0^{-1}\varkappa + {}^{17}/_{45}b_0^{-1} + {}^1/_3 b_0^{-2}) + k_0^4\,(-{}^{271}/_{1575} +$$
$$+ {}^{16}/_{45}b_0^{-1}\varkappa + {}^{92}/_{45}b_0^{-1} - {}^4/_9 b_0^{-2})] - {}^1/_{12}\lambda_0^{-1}\varkappa_k^2 b_0^{-1}\}\,(1 - 2\lambda_0\varkappa_k^{-1})^{-3}$$

Taking account of the representation of the operator Q in the form (1.16) for large values of k ($k \approx \varepsilon^{-1}$), the first equation (1.20) can be given the form

$$[-{}^4/_3 D_1^8 + {}^8/_5 D_1^{10} - {}^{808}/_{945}D_1^{12} + \ldots] + \varepsilon^2\,[2b_0 p_3^4 + ({}^{16}/_3 - 4b_0)D_1^2 p_3^4 +$$
$$+ D_1^4(-{}^8/_3 p_3^2 - {}^{32}/_5 p_3^4 + {}^{52}/_{15}b_0 p_3^4) + \ldots] + \ldots = 0$$
$$(D_1^2 = p_3^2 - k_1^2,\ k_1 = \varepsilon k,\ p_3 = \varepsilon\lambda_{ki}) \qquad (2.11)$$

We hence find

$$\lambda_{ki} = \varepsilon^{-1}(p_{30} + \varepsilon^{1/2}p_{31} + \varepsilon p_{32} + \varepsilon^{3/2}p_{33} + \ldots), \qquad p_{30}^2 - k_1^2 = 0$$
$$p_{31}^4 - {}^3/_{32}b_0 = 0, \qquad p_{32} = p_{31}^2\,[{}^1/_2 p_{30}^{-1} + p_{30}\,({}^4/_3 b_0^{-1} - {}^2/_5)] \quad (2.12)$$
$$p_{33} = p_{31}^{-1}\,[{}^1/_8 - {}^3/_{40}b_0 + p_{30}^2\,(-{}^1/_{12}b_0^{-1} + {}^1/_{20} + {}^{41}/_{8400}b_0)]$$

Formulas resulting from (1.16) and yielding a good approximation for roots of the type (2.7), (2.8) when $k \lesssim \varepsilon^{-1}$ are presented below

$$\lambda_{ki} = (2\varepsilon)^{-1}\nu_0 + 2\varepsilon\nu_1 + (2\varepsilon)^3\nu_2 + \ldots, \qquad k_* = 2\varepsilon k$$
$$(\sin^2 x_1 - x_1^2)\,/\,x_1^4 = 0, \qquad \nu_0^2 = k_*^2 + x_1^2 \qquad (2.13)$$
$$\nu_1 = (\sin 2x_1 - 2x_1)^{-1}\,[{}^1/_3 x_1^3\nu_0^{-1} + {}^1/_3 x_1\nu_0 - \nu_0^3\,(2b_0 x_1^{-3} + {}^2/_3 x_1^{-1})] +$$
$$+ \nu_0\,[(2 - 2\varkappa)\,x_1^{-2} - {}^1/_{12}] + \nu_0^3\,[(-{}^{17}/_8 + {}^5/_2\varkappa)\,x_1^{-4} + {}^1/_{24}x_1^{-2}]$$
$$\lambda_{ki} = (2\varepsilon)^{-1}\mu_0 + 2\varepsilon\mu_1 + (2\varepsilon)^3\mu_2 + \ldots, \qquad \mu_0^2 = k_*^2 + x_0^2 \qquad (2.14)$$
$$\mu_1 = \mu_0^3\,[({}^{43}/_8 - 2\varkappa)\,x_0^{-4} + {}^1/_{24}x_0^{-2}] + \mu_0\,[(-{}^7/_2 + 2\varkappa)\,x_0^{-2} - {}^1/_{12}]$$

Finally, for very large λ_{ki} and k ($k \gg \varepsilon^{-1}$) the roots λ_{ki} should be sought from the asymptotic equation

$$(\mathrm{sh}\,\gamma_1\,\mathrm{sh}\,\gamma_2)^{-1/2}\,(1 - e^{2\varepsilon}k^2\,/\,\lambda_{ki}^2)^2\,(1 - e^{-2\varepsilon}k^2\,/\,\lambda_{ik}^2)^2\,Q^{**}\,\mathrm{sh}\,\theta + O(\zeta_1^{-1}) = 0$$
$$\theta = k\,(\mathrm{th}\gamma_1 - \mathrm{th}\,\gamma_2 - \gamma_1 + \gamma_2), \qquad \mathrm{ch}\,\gamma_1 = e^{\varepsilon}k\,/\,\lambda_{ki}, \qquad \mathrm{ch}\gamma_2 = e^{-\varepsilon}k\,/\,\lambda_{ki}$$
$$Q^{**} = 2\lambda_{ki}^2\,[\mathrm{ch}\,2\varepsilon - \mathrm{ch}\,(\gamma_2 - \gamma_1)] + \mathrm{sh}^2\theta\,(1 - \mathrm{cth}^2\gamma_1)\,(1 - \mathrm{cth}^2\gamma_2) \quad (2.15)$$
$$\zeta_1 = \max\{\lambda_{ki}, k\}$$

which results from the representation of the operator Q in the form (1.15), and from asymptotic expansions of the quantities L_{kj} for large and complex k and λ_{ki} (see e.g. [7]).

$$L_{00} = (\mathrm{sh}\gamma_1 \mathrm{sh}\gamma_2)^{-1/2} \{\mathrm{sh}\theta + k^{-1}\, {}^1/_2 \,\mathrm{ch}\,\theta A_1 + \ldots\}$$
$$L_{11} = (\mathrm{sh}\,\gamma_1\,\mathrm{sh}\,\gamma_2)^{1/2} - \{\mathrm{sh}\,\theta + k^{-1}\, {}^1/_2\, \mathrm{ch}\,\theta\, (\mathrm{cth}\,\gamma_1 - \mathrm{cth}\gamma_2 - \mathrm{cth}^3\gamma_1 + $$
$$+ \mathrm{cth}^3\gamma_2 - A_1) + \ldots\}$$

$$L_{01} = (\mathrm{sh}\,\gamma_2 \,/\, \mathrm{sh}\gamma_1)^{1/2} \{- \mathrm{ch}\,\theta + k^{-1}\, {}^1/_2\, \mathrm{sh}\theta\, (\mathrm{cht}^3\gamma_2 - \mathrm{cth}\gamma_2 - A_1) + \ldots\}$$
$$L_{10} = (\mathrm{sh}\,\gamma_1 \,/\, \mathrm{sh}\,\gamma_2)^{1/2} \{\mathrm{ch}\,\theta + k^{-1}\, {}^1/_2\, \mathrm{sh}\,\theta\, (\mathrm{cth}^3\gamma_1 - \mathrm{cth}\,\gamma_1 + A_1) + \ldots\}$$
$$(A_1 = {}^1/_4\, (\mathrm{cth}\,\gamma_1 - \mathrm{cth}\,\gamma_2) - {}^5/_{12}\, (\mathrm{cth}^3\gamma_1 - \mathrm{cth}^3\gamma_2)) \qquad (2.16)$$

We conclude from (2.15) that the asymptotic equalities

$$\lambda_{ki} \approx \pm\, ke^{\varepsilon}, \quad \lambda_{ki} \approx \pm\, ke^{-\varepsilon} \qquad \text{for} \qquad k \to \infty \qquad (2.17)$$

correspond to the eight roots λ_{ki}, and the principal parts of the remaining roots are found from the equations

$$\theta = 2\lambda_{k_m}\, \mathrm{sh}\, \varepsilon i\, (-1 + {}^1/_2\gamma_0^2 + \ldots) = im\pi \qquad (i = \sqrt{-1},\ m = 1, 2, \ldots)$$
$$Q^{**} = (2\lambda_{ki}\, \mathrm{sh}\, \varepsilon)^2\, (1 + \gamma_0^2 + \ldots) + \mathrm{sh}^2\theta\, (1 + 2\gamma_0^2\, \mathrm{ch}\, 2\varepsilon + \ldots) = 0$$
$$(\gamma_0 = k \,/\, \lambda_{ki}) \qquad (2.18)$$

which are obtained from (2.15) by using expansions valid for $|\gamma_0 e^{\varepsilon}| \leqslant 1$.

The analysis carried out shows that the first characteristic equation (1.20) contains three groups of roots.

The first group contains two exact quadruple roots $\lambda_{0i} = 0$ for $k = 0$, and $\lambda_{1i} = 0$ for $k = 1$.

The second group consists of eight roots determined by (2.2), (2.5), (2.10), (2.12), (2.17). The order of the moduli of these roots hence depends on k. If $k < \varepsilon^{-1/2}$, then the moduli of four of them are commensurate with $\varepsilon^{1/2}k^2$ (small roots), and the four other roots are commensurate in absolute value with $\varepsilon^{-1/2}$ (large roots). For $k \gtrsim \varepsilon^{-1/2}$ all eight roots are commensurate with k in absolute value. In the case of large and very large k ($k \approx \varepsilon^{-1}$ and $k \gg \varepsilon^{-1}$) the asymptotic equalities $\lambda_{ki} \approx \pm\, k$ and $\lambda_{ki} \approx \pm$ $\pm\, k \exp\, (\pm\, \varepsilon)$ are valid, respectively.

The third group consists of a countable set of roots determined by (2.7), (2.8), (2.13), (2.14), (2.18), and growing as $1 / \varepsilon$ as $\varepsilon \to 0$.

3. Analysis of the state of stress and strain corresponding to each group of roots. Group (1). The stress functions (3.1)

$$\Psi_1^*\, (\zeta, \varphi) = T_{-1}\zeta + T_0\zeta^2 + T_1\zeta^3 + (N_{1,2} + N_{1,2}{}^*\zeta + M_{1,2}\zeta^2 + M_{1,2}{}^*\zeta^3)\, e^{i\varphi}$$

correspond to the quadruple roots $\lambda_{0i} = 0$ and $\lambda_{1i} = 0$, where $T_{-1}, \ldots, M_{1,2}{}^*$ are arbitrary constants, and the subscript 1, 2 provisionally denotes $a_{1,2}e^{i\varphi} \equiv a_1 \cos\varphi +$ $+\, a_2 \sin\varphi$.

Substituting (3.1) into (1.6), and moreover (1.6) into (1.1–3), we obtain

$$u = -\, {}^1/_2 R_1 a_2 \rho_1 T_0, \qquad v = 0, \qquad w = -\, R_1 \zeta_1 T_0$$
$$\sigma_z = G a_0 T_0, \qquad \sigma_r = \sigma_\varphi = \tau_{r\varphi} = \tau_{rz} = \tau_{z\varphi} = 0 \qquad (3.2)$$
$$u = 0, \qquad v = R_1 \zeta_1 \rho_1 T_1, \qquad w = 0$$
$$\tau_{z\varphi} = G \rho_1 T_1, \qquad \sigma_r = \sigma_z = \sigma_\varphi = \tau_{r\varphi} = \tau_{rz} = 0 \qquad (3.3)$$

$$u = R_3\varepsilon^{-1}\{[\varkappa\Delta^2 + b_0\lambda^2] + \varepsilon[-(c_1 + a_0)\Delta^2\lambda^2 - ({}^7/_3 + {}^{13}/_6\varkappa)\Delta^4 + t(b_2\lambda^2 + {}^1/_3 a_2\Delta^6) + {}^1/_2 t^2\langle(a_0 - b_0)\Delta^2\lambda^2 - a_2\Delta^4\rangle] + \ldots\}\,\partial\Psi_1/\partial\zeta \tag{3.9}$$

$$w = R_3\varepsilon^{-1}\{[\lambda^2 B_0(t) + \varkappa\lambda^2 + {}^4/_3\Delta^6 - t\varkappa\Delta^2\lambda^2] + \varepsilon[\lambda^2 B_1(t) + {}^8/_3\Delta^4 - {}^1/_3 b_5\lambda^4 - ({}^8/_3 + {}^7/_6\varkappa)\Delta^2\lambda^2 - {}^{62}/_{45}\Delta^8 + t({}^1/_3 + {}^{13}/_6\varkappa)\Delta^4\lambda^2 - t^2\langle{}^1/_2\varkappa(\Delta^2\lambda^2 + \lambda^4) + {}^2/_3\Delta^8\rangle + {}^1/_6 t^3 a_6\Delta^4\lambda^2] + \ldots\}\Psi_1$$

$$v = R_3\{[B_0(t) - \varkappa - t\varkappa\Delta^2] + \varepsilon[B_1(t) + {}^1/_3 b_3\lambda^2 - ({}^4/_3 - {}^{23}/_6\varkappa)\Delta^2 - t\langle\varkappa + (1 - {}^5/_2\varkappa)\Delta^4\rangle - t^2({}^1/_2\varkappa\Delta^2 + b_1\lambda^2) + {}^1/_6 t^3 a_6\Delta^4] + \ldots\}\,\partial^2\Psi_1/\partial\zeta\partial\varphi$$

$$\tau_{z\varphi} = 2G\varepsilon^{-1}\{[\lambda^2 B_0(t) + {}^2/_3\Delta^6 - t\varkappa\Delta^2\lambda^2] + \varepsilon[\lambda^2 B_1(t) + {}^4/_3\Delta^4 - {}^1/_6\varkappa\lambda^4 - {}^{31}/_{45}\Delta^8 - (2 - {}^4/_3\varkappa)\Delta^2\lambda^2 + t\langle({}^1/_3 + {}^{13}/_6\varkappa)\Delta^4\lambda^2 - {}^2/_3\Delta^6 - \varkappa\lambda^2\rangle - t^2({}^1/_2\varkappa\lambda^4 + {}^1/_3\Delta^8) + {}^1/_6 t^3 a_6\Delta^4\lambda^2] + \ldots\}\,\partial\Psi_1/\partial\varphi$$

$$\sigma_z = 2G\varepsilon^{-1}\{[\lambda^2 B_0(t) - {}^1/_3 a_{-2}\Delta^6 - b_0\lambda^2 + t\langle a_2\Delta^4 + (b_0 - 2a_2)\Delta^2\lambda^2\rangle] + \varepsilon[\lambda^2 B_1(t) + {}^1/_3 b_{-1}\lambda^4 + ({}^3/_4\varkappa^2 - {}^1/_3 a_{26})\Delta^2\lambda^2 + (6 - {}^5/_3\varkappa)\Delta^4 - (c_2 + {}^{26}/_{45})\Delta^8 + t\langle a_2(\Delta^2 - \lambda^2 - {}^5/_2\Delta^6) + ({}^1/_3 a_{-17} - {}^{11}/_6 b_{-4})\Delta^4\lambda^2\rangle + t^2({}^1/_2 b_0\Delta^2\lambda^2 - {}^1/_2\varkappa\lambda^4 + {}^1/_6 a_{-2}\Delta^8) + t^3\langle-{}^1/_6 a_2\Delta^6 + ({}^1/_2 a_2 - c_3)\Delta^4\lambda^2\rangle] + \ldots\}\partial\Psi_1/\partial\zeta \tag{3.10}$$

$$\sigma_\varphi = 2G\varepsilon^{-1}\{[-\lambda^2 B_0(t) - {}^2/_3\Delta^6 - 2t(\Delta^4 - 2\Delta^2\lambda^2)] + \varepsilon[-\lambda^2 B_1(t) + {}^1/_3 b_1\lambda^4 - {}^1/_3\Delta^4 + {}^4/_5\Delta^8 - {}^1/_3\varkappa\Delta^2\lambda^2 + t(2\lambda^2 - 2\Delta^2 + {}^{14}/_3\Delta^6 - {}^1/_2 a_{18}\Delta^4\lambda^2) + t^2(\Delta^4 - 2\Delta^2\lambda^2 + {}^1/_2\varkappa\lambda^4) + t^3({}^2/_3\Delta^6 + {}^1/_6 a_{-6}\Delta^4\lambda^2)] + \ldots\}\partial\Psi_1/\partial\zeta$$

$$\sigma_r = 2G(t^2 - 1)\{[2\Delta^2\lambda^2 + {}^1/_3\Delta^8 - \Delta^4 - t({}^1/_3\Delta^6 + {}^1/_6 a_0\Delta^4\lambda^2)] + \varepsilon[\lambda^2 - \Delta^2 + {}^{17}/_6\Delta^6 - {}^1/_3 a_{10}\Delta^4\lambda^2 + {}^1/_3 a_{-4}\Delta^2\lambda^4 - {}^{31}/_{90}\Delta^{10} + t({}^1/_3\Delta^4 - {}^1/_6 a_{10}\Delta^2\lambda^2 - {}^1/_3 b_{-2}\lambda^4 + {}^7/_{10}\Delta^8 - c_0\Delta^6\lambda^2) + t^2({}^1/_2\Delta^6 + {}^1/_6 a_{-6}\Delta^4\lambda^2 - {}^1/_3 a_0\Delta^2\lambda^4 - {}^1/_{18}\Delta^{10}) + t^3({}^1/_{30}\Delta^8 + {}^1/_{60}a_0\Delta^6\lambda^2)] + \ldots\}\,\partial\Psi_1/\partial\zeta$$

$$\tau_{r\varphi} = 2G(t^2 - 1)\{B_2(t) + \varepsilon[B_3(t) + \Delta^2 + {}^1/_2 a_2\lambda^2 - {}^5/_2\Delta^6 - {}^1/_6 t^2\Delta^6 + t(2\Delta^2\lambda^2 - \Delta^4)] + \ldots\}\partial^2\Psi_1/\partial\zeta\,\partial\varphi$$

$$\tau_{rz} = 2G(t^2 - 1)\varepsilon^{-1}\{\lambda^2 B_2(t) + \varepsilon[\lambda^2 B_3(t) + \Delta^2\lambda^2 - \lambda^4 + {}^2/_3\Delta^8 - ({}^{25}/_6 - {}^1/_3\varkappa)\Delta^6\lambda^2 + t\langle({}^2/_3 a_3 - {}^1/_2 b_0)\Delta^2\lambda^4 - {}^1/_3 a_3\Delta^4\lambda^2 + {}^1/_9\Delta^{10}\rangle - {}^1/_6 t^2\Delta^6\lambda^2] + \ldots\}\Psi_1 \tag{3.11}$$

$$B_0(t) = {}^1/_3 a_0\Delta^4 - tb_0\lambda^2,\quad B_2(t) = \Delta^4 + {}^1/_2 a_0\Delta^2\lambda^2,\quad B_1(t) = \Delta^6(c_2 - {}^1/_6 t^2 a_2) + t\Delta^2\lambda^2(c_1 + c_3 t^2)$$

$$B_3(t) = -{}^{13}/_{12}a_0\Delta^4\lambda^2 + {}^1/_3 t(b_0\lambda^4 + {}^1/_3\Delta^8) - {}^1/_{12}t^2 a_0\Delta^4\lambda^2$$

$$c_1 = 2b_0 - c_3,\quad c_2 = {}^{77}/_{45} - {}^{41}/_{90}\varkappa,\quad c_3 = {}^1/_6(a_0 + b_0),\quad c_0 = {}^{53}/_{180}\varkappa - {}^{43}/_{45}$$

It follows from (3.9), (3.10) and (2.5) that the quantities $u, v, \ldots, \tau_{r\varphi}$ corresponding to the large roots and satisfying the relationships

$$|u|, |\sigma_z|, |\sigma_\varphi| \approx \varepsilon^{-3/2}, \qquad |\tau_{z\varphi}| \approx \varepsilon^{-1}k$$
$$|w|, |\tau_{rz}| \approx \varepsilon^{-1}, \qquad |v|, |\tau_{r\varphi}| \approx \varepsilon^{-1/2}k, \ |\sigma_r| \approx \varepsilon^{-1/2} \tag{3.12}$$

decrease as $\exp(-\varepsilon^{-1/2}p^{**}s_1)$ ($\operatorname{Re} p^{**} > 0$) with advancement into the domain occupied by the shell. Thus, the solutions corresponding to large roots are edge effects, whose damping zones will be narrower, the smaller the ε.

In the case of roots defined by (2.10) and commensurate with k $(\varepsilon^{-1/2} \lesssim k < \varepsilon^{-1})$ the following estimates are obtained from (3.9), (3.10):

$$|u| \approx k^3, \qquad |v|, |w| \approx \varepsilon k^4, \qquad |\tau_{z\varphi}|, |\sigma_z|, |\sigma_\varphi| \approx \varepsilon k^5$$
$$|\sigma_r| \approx \varepsilon^2 k^5, \qquad |\tau_{r\varphi}|, |\tau_{rz}| \approx \varepsilon^{3/2} k^5 \tag{3.13}$$

Hence, all the characteristics of the state of stress and strain decrease as $\exp(-ks_1)$. Therefore, as k increases its corresponding homogeneous solutions become damped all the more rapidly. For $k \approx \varepsilon^{-1}$ the following solutions correspond to roots defined by (2.12):

$$u = R_3\varepsilon^{-2}\{b_0p_3^2 + \varepsilon^{1/2}[\varkappa\Lambda^2 - (c_1 + a_0)\Lambda^2p_3^2 + {}^1/_3 ta_2\Lambda^6 + {}^1/_2 t^2(a_0 - b_0)\Lambda^2p_3^2] + \ldots\}\,\partial\Psi_1/\partial\zeta$$

$$v = R_3\varepsilon^{-1}\{B_0^*(t) + \varepsilon^{1/2}[B_1^*(t) - t\varkappa\Lambda^2] + \ldots\}\,\partial^2\Psi_1/\partial\zeta\partial\varphi \tag{3.14}$$

$$w = R_3\varepsilon^{-3}\{p_3^2B_0^*(t) + \varepsilon^{1/2}[p_3^2B_1^*(t) + {}^4/_3\Lambda^6 - t\varkappa\Lambda^2p_3^2] + \ldots\}\,\Psi_1$$

$$\sigma_z = 2G\varepsilon^{-3}\{p_3^2B_0^*(t) + \varepsilon^{1/2}[p_3^2B_1^*(t) - {}^1/_3 a_{-2}\Lambda^6 + t\Lambda^2p_3^2(b_0 - 2a_2)] + \ldots\}\,\partial\Psi_1/\partial\zeta$$

$$\sigma_\varphi = 2G\varepsilon^{-3}\{-p_3^2B_0^*(t) + \varepsilon^{1/2}[-p_3^2B_1^*(t) - {}^2/_3\Lambda^6 + 4t\Lambda^2p_3^2] + \ldots\}\,\partial\Psi_1/\partial\zeta$$

$$\tau_{z\varphi} = 2G\varepsilon^{-3}\{p_3^2B_0^*(t) + \varepsilon^{1/2}[p_3^2B_1^*(t) + {}^2/_3\Lambda^6 - t\varkappa\Lambda^2p_3^2] + \ldots\}\,\partial\Psi_1/\partial\varphi$$
$$\tau_{rz} = 2G(t^2 - 1)\,\varepsilon^{-7/2}\{[{}^1/_2 a_0\Lambda^2p_3^4] + \varepsilon^{1/2}[p_3^2B_2^*(t)] + \ldots\}\,\Psi_1 \tag{3.15}$$

$$\tau_{r\varphi} = 2G(t^2 - 1)\,\varepsilon^{-3/2}\{[{}^1/_2 a_0\Lambda^2p_3^2] + \varepsilon^{1/2}[B_2^*(t)] + \ldots\}\,\partial^2\Psi_1/\partial\zeta\partial\varphi$$

$$\sigma_r = 2G(t^2 - 1)\varepsilon^{-2}\{[{}^1/_3\Lambda^8 - {}^1/_6 ta_0\Lambda^4p_3^2] + \varepsilon^{1/2}[\Lambda^2(2p_3^2 + {}^1/_3 a_{-4}p_3^4 - {}^{31}/_{90}\Lambda^8) + t\Lambda^6(c_0p_3^2 - {}^1/_3) - {}^1/_3 t^2\Lambda^2(a_0p_3^4 + {}^1/_6\Lambda^8) + {}^1/_{60} t^3a_0\Lambda^6p_3^2] + \ldots\}\,\partial\Psi_1/\partial\zeta$$

$$B_0^*(t) = {}^1/_3 a_0\Lambda^4 - tb_0p_3^2, \qquad \Lambda^2 = \varepsilon^{-1/2}D_1^2$$
$$B_1^*(t) = \Lambda^6(c_2 - {}^1/_6 t^2a_2) + t\Lambda^2p_3^2(c_1 + c_3t^2) \tag{3.16}$$
$$B_2^*(t) = \Lambda^4(1 - {}^{13}/_{12} a_0p_3^2) + {}^1/_3 t(b_0p_3^4 + {}^1/_3\Lambda^8) - {}^1/_{12} t^2a_0\Lambda^4p_3^2$$

Analyzing the estimating formulas (3.13), which are also suitable for the relationships (3.14), (3.15), it can be noted that for large k $(k \gtrsim \varepsilon^{-1})$, the homogeneous solutions (3.14), (3.15) are governed, in a first approximation, by the quantities v, w, σ_z, σ_φ, $\tau_{z\varphi}$, i.e. correspond to some plane state of stress.

Group (3). If $k \lesssim \varepsilon^{-1}$, then by expanding the solutions of this group in powers of the small parameter ε and limiting oneself to the first member of the expansion, we find the following asymptotic expressions:

$$u = \varepsilon R_3D_0^2\{\sin t_1D_0\,[(1 - \varkappa - t_1)D_0\sin^2 D_0 - {}^1/_2\, t_1D_0^2\sin 2D_0] + \cos t_1D_0\,[(t_1D_0^2 - \varkappa)\sin^2 D_0 - {}^1/_2\varkappa D_0\sin 2D_0]\}\,\Psi_1 \tag{3.17}$$
$$v = 2\varepsilon^2R_3P_1(t_1)\,\partial\Psi_1/\partial\varphi, \qquad w = 2\varepsilon^2R_3P_1(t_1)\,\partial\Psi_1/\partial\zeta$$

$$\sigma_r = GD_0^3 \{\sin t_1D_0 [(1 - t_1D_0^2) \sin^2 D_0 + {}^1/_2 D_0 \sin 2D_0] - \tag{3.18}$$
$$- \cos t_1D_0 [t_1D_0 \sin^2 D_0 + {}^1/_2 t_1 D_0^2 \sin 2D_0]\} \Psi_1$$
$$\tau_{r\varphi} = \varepsilon GP_2(t_1)\, \partial\Psi_1 / \partial\varphi, \quad \tau_{rz} = \varepsilon GP_2(t_1) \partial\Psi_1 / \partial\zeta,$$
$$\tau_{z\varphi} = 4\varepsilon^2 GP_1(t_1)\, \partial^2\Psi_1 / \partial\zeta\partial\varphi$$
$$\sigma_z = G(\lambda_*^2 P_1(t_1) - a_2P_3(t_1)\Psi_1, \qquad \sigma_\varphi = -G(k_*^2P_1(t_1) + a_2P_3(t_1))\Psi_1$$

$$P_1(t_1) = D_0 \{\sin t_1D_0 [(t_1D_0^2 - 1 + \varkappa) \sin^2 D_0 + {}^1/_2 D_0 (\varkappa - 1) \sin 2D_0] +$$
$$+ \cos t_1D_0 [D_0(t_1 - \varkappa) \sin^2 D_0 + {}^1/_2 t_1D_0^2 \sin 2D_0]\}$$
$$P_2(t_1) = D_0^3\{\sin t_1D_0 [2(1 - t_1) \sin^2 D_0 - t_1D_0 \sin 2D_0] +$$
$$+ 2t_1D_0 \sin^2 D_0 \cos t_1D_0\} \tag{3.19}$$
$$P_3(t_1) = D_0^3 \{\sin t_1D_0 (\sin^2 D_0 + {}^1/_2 D_0 \sin 2D_0) - D_0 \sin^2 D_0 \cos t_1D_0\}$$
$$D_0^2 = \lambda_*^2 - k_*^2, \quad \lambda_* = 2\varepsilon\lambda_{ki}, \quad k_* = 2\varepsilon k, \quad t_1 = (2\varepsilon)^{-1} \ln \rho_1$$

and the roots λ_{ki} are found by means of (2.7), (2.13).

In the case of roots defined by (2.8), (2.14), we find the following expressions:

$$u = 0, \quad v = 2\varepsilon^2 R_3\lambda_*^2 \cos t_1D_0 \partial\Psi_1 / \partial\varphi, \quad w = 2\varepsilon^2 R_3 k_*^2 \cos t_1D_0 \partial\Psi_1 / \partial\zeta \tag{3.20}$$
$$\tau_{z\varphi} = 2\varepsilon^2 G(k_*^2 + \lambda_*^2) \cos t_1D_0 \partial^2\Psi_1 / \partial\zeta\partial\varphi, \quad \sigma_z = -\sigma_\varphi = G\lambda_*^2 k_*^2 \cos t_1D_0\Psi_1$$
$$\tau_{rz} = -\varepsilon GD_0k_*^2 \sin t_1D_0 \partial\Psi_1 / \partial\zeta, \quad \tau_{r\varphi} = -\varepsilon GD_0\lambda_*^2 \sin t_1D_0\, \partial\Psi_1 / \partial\varphi$$
$$\sigma_r = 0 \tag{3.21}$$

It follows from (3.17), (3.18) and (3.20), (3.21) that for small ε and $k \lesssim \varepsilon^{-1}$, the displacements and stresses corresponding to roots of the third group are subject to the relationships

$$|u|, |w| \approx \varepsilon, \quad |v| \approx \varepsilon^2 k, \quad |\sigma_r|, |\sigma_z|, |\sigma_\varphi|, |\tau_{rz}| \approx 1, \quad |\tau_{r\varphi}|_z |\tau_{z\varphi}| \approx \varepsilon k \tag{3.22}$$
$$|v| \approx \varepsilon^2 k, \quad |w| \approx \varepsilon^3 k^2, \quad |\sigma_z|, |\sigma_\varphi|, |\tau_{rz}| \approx \varepsilon^2 k^2, \quad |\tau_{z\varphi}|, |\tau_{r\varphi}| \approx \varepsilon k \tag{3.23}$$

and decrease as $\exp(-\varepsilon^{-1}p^{***}s_1)$ ($\mathrm{Re}\, p^{***} > 0$) with recession from the boundary Γ_2. It is important to emphasize that the relationships (3.17)–(3.21) actually agree with the homogeneous solutions obtained in plate theory [6].

All the above affords a foundation to conclude that the edge effects of applied shell theory correspond to the second group of solutions. The third group of solutions yields the boundary layers which are generally absent in Kirchhoff-Love theory.

4. Analysis of roots of the characteristic equation of an open cylindrical shell. Utilizing the representation of the operator Q in the form (1.14), we easily establish that the second equation (1.20) reduces for $n_m = 0$ to

$$(k_{ni} + k_{ni}^3)\sin 2\varepsilon k_{ni} (\sin^2 2\varepsilon k_{ni} - k_{ni}^2 \mathrm{sh}^2 2\varepsilon) = 0 \tag{4.1}$$

This latter contains two groups of roots:

1) Quadruple roots $k_{0i} = 0$ and double roots $k_{0i} = \pm i$;

2) A countable set of roots growing as $1/\varepsilon$ as $\varepsilon \to 0$ and defined by the formulas

$$k_{0m} = (2\varepsilon)^{-1} m\pi \quad (m = 1,2,\ldots), \qquad k_{0i} = (2\varepsilon)^{-1}k_{00} + 2\varepsilon k_{11} + (2\varepsilon)^3 k_{22} + \ldots$$
$$(\sin^2 k_{00} - k_{00}^2) / k_{00}^4 = 0, \qquad k_{11} = 1/3 k_{00}^2 (\sin 2k_{00} - 2k_{00})^{-1} \tag{4.2}$$

$$k_{22} = 2/45\ [k_{00}^3 (\sin 2k_{00} - 2k_{00})^{-2} + k_{00}^6 (\sin 2k_{00} - 2k_{00})^{-3}] \qquad \text{(cont.)}$$

We apply a method expounded in the Gol'denveizer monograph [8] to investigate the roots of the characteristic equation (1.20) when $n_m \neq 0$, $\varepsilon \to 0$. In the case of small n_m ($n_m \lesssim \varepsilon^{1/2}$), we obtain by making the change $n_m = \varepsilon^{1/2} n_0$ in (1.20)

$$[2b_0 n_0^4 - {}^4/_3 k_{ni}^4 (1 + k_{ni}^2)^2] + \varepsilon[{}^{16}/_3 k_{ni}^2 (1 + k_{ni}^2)^2 n_0^2] + \varepsilon^2 [4b_0 n_0^4 (2 - k_{ni}^2) - \\ -8k_{ni}^2 (1 + k_{ni}^2) n_0^4 - {}^8/_{45} k_{ni}^4 (1 + k_{ni}^2)^2 (4 - 9k_{ni}^2)] + \ldots = 0 \qquad (4.3)$$

Hence, as $\varepsilon \to 0$ the following asymptotic expansion results

$$k_{ni} = k_{ni0} + \varepsilon k_{ni1} + \varepsilon^2 k_{ni2} + \cdots, \qquad k_{ni0}^4 (1 + k_{ni0}^2)^2 - {}^3/_2 b_0 n_0^4 = 0 \qquad (4.4)$$

$$k_{ni1} = \frac{n_0^2 (1 + k_{ni0}^2)}{k_{ni0} (1 + 2k_{ni0}^2)}$$

$$k_{ni2} = \frac{n_0^4}{k_{ni0}^3} \left[\frac{b_0 (13 - 3k_{ni0}^2)}{10 (1 + k_{ni0}^2)(1 + 2k_{ni0}^2)} + \frac{1 + 3k_{ni0}^2 + 3k_{ni0}^4 - 2k_{ni0}^6}{2 (1 + 2k_{ni0}^2)^3} \right]$$

For medium values of n_m ($\varepsilon^{1/2} < n_m < \varepsilon^{-1/2}$), the substitution $k_{ni} = \varepsilon^{-1/4} k_2$ reduces (1.20) to

$$(2b_0 n_m^4 - {}^4/_3 k_2^8) + \varepsilon^{1/2} k_2^6 (-{}^8/_3 + {}^{16}/_3\, n_m^2) + \varepsilon k_2^4 (-{}^4/_3 + {}^{32}/_3 n_m^2 - 8n_m^4) + \\ + \varepsilon^{3/2} k_2^2 ({}^{16}/_3 n_m^2 - 4b_0 n_m^4 - 8n_m^4 + {}^{16}/_3\, n_m^6 + {}^8/_5 k_2^8) + \ldots = 0 \qquad (4.5)$$

We hence find

$$k_{ni} = \varepsilon^{-1/4} (k_{20} + \varepsilon^{1/2} k_{21} + \varepsilon k_{22} + \varepsilon^{3/2} k_{23} + \ldots), \qquad k^8_{20} - {}^3/_2 b_0 n_m^4 = 0$$
$$k_{21} = k_{20}^{-1} (-{}^1/_4 + {}^1/_2 n_m^2), \qquad k_{22} = k_{20}^{-3} ({}^1/_{32} + {}^3/_8 n_m^2 - {}^1/_8 n_m^4) \qquad (4.6)$$
$$k_{23} = k_{20}^{-5} [{}^1/_{128} + {}^5/_{64} n_m^2 + ({}^9/_{32} - {}^3/_{20} b_0) n_m^4 + {}^1/_{16} n_m^6]$$

In the case of large values of n_m ($\varepsilon^{-1/2} \lesssim n_m < \varepsilon^{-1}$), we apply the substitution $n_m = \varepsilon^{-1/2} n_1$ and $k_{ni} = \varepsilon^{-1/2} k_3$, we represent (1.20) in the form (2.9), wherein we put $\Delta^2 = k_3^2 - n_1^2$, $\lambda^2 = -n_1^2$. Now, expanding k_3 in a series in ε, we obtain

$$k_{ni} = \varepsilon^{-1/2} (k_{30} + \varepsilon k_{31} + \varepsilon^2 k_{32} + \ldots), \quad k_{30}^2 - \varkappa_k n_1 - n_1^2 = 0$$
$$k_{31} = k_{30}^{-1} [-{}^1/_4 + {}^1/_4 \varkappa_k^{-1} n_1 + n_1^2 ({}^1/_2 - {}^3/_{20} b_0) \varkappa_k^{-2}], \quad \varkappa_k^4 - {}^3/_2 b_0 = 0$$
$$k_{32} = k_{30}^{-3} [{}^1/_{16} b_0^{-1} - {}^1/_{32} + {}^1/_{16} \varkappa_k^{-1} (1 + b_0^{-1}) n_1 + \varkappa_k^2 ({}^3/_{16} b_0^{-1} + {}^{11}/_{24}) n_1^2 + \varkappa_k ({}^5/_{24} \\ b_0^{-1} + {}^{19}/_{24} + {}^1/_{15} \varkappa) n_1^3 + (-{}^1/_4 b_0^{-1} + {}^{29}/_{60} + {}^1/_{15} \varkappa + {}^{19}/_{84000} b_0)\, n_1^4 + \\ + \varkappa_k^{-1} (-{}^1/_6 b_0^{-1} + {}^1/_{10} + {}^{41}/_{4200} b_0) n_1^5] \qquad (4.7)$$

Finally, for $n_m \approx \varepsilon^{-1}$, equation (1.20) can be written in the form (2.11) by virtue of (1.16), by putting $D_1^2 = k_4^2 - n_2^2$, $p_3^2 = -n_2^2$, $k_4 = \varepsilon k_{ni}$, $n_2 = \varepsilon n_m$ (4.8)

Taking account of (4.8) it follows from (2.11) that:

$$k_{ni} = \varepsilon^{-1} (k_{40} + \varepsilon^{1/2} k_{41} + \varepsilon k_{42} + \varepsilon^{3/2} k_{43} + \ldots), \qquad k_{40}^2 - n_2^2 = 0$$
$$k_{41}^4 - {}^3/_{32} b_0 = 0, \qquad k_{42} = k_{40}^2 [-{}^1/_2 n_2^{-1} + n_2 ({}^1/_3 b_0^{-1} - {}^2/_5)]$$
$$k_{43} = k_{40}^{-1} [{}^3/_{64} b_0 n_2^{-2} + {}^3/_{80} b_0 + n_2^2 (-{}^1/_{12} b_0^{-1} + {}^1/_{20} + {}^{41}/_{8400} b_0)] \qquad (4.9)$$

Furthermore, taking account of the representation of the operator Q in the form (1.16), it is easy to establish that the second characteristic equation (1.20), in addition to the eight roots found above, also has a countable set of other roots for which $k_{ni}\varepsilon \to \text{const}$ as $\varepsilon \to 0$. For $n_m \lesssim \varepsilon^{-1}$, the asymptotic of the mentioned roots can be obtained by utilizing the expansions

$$k_{ni} = (2\varepsilon)^{-1}\sigma_0 + 2\varepsilon\sigma_1 + (2\varepsilon)^3\sigma_2 + \ldots$$
$$x_0^{-1}\sin x_0 = 0, \quad \sigma_0^2 = n_*^2 + x_0^2, \quad n_* = 2\varepsilon n_m \tag{4.10}$$
$$\sigma_1 = \sigma_0^{-1} n_*^2 \{n_*^2[(^{43}/_8 - 2\varkappa)x_0^{-4} + {}^1/_{24}x_0^{-2}] + (-{}^7/_{12} + 2\varkappa)x_0^{-2} - {}^1/_{12}\}$$
$$k_{ni} = (2\varepsilon)^{-1}\omega_0 + 2\varepsilon\omega_1 + (2\varepsilon)^3\omega_2 + \ldots$$
$$(\sin^2 x_1 - x_1^2) / x_1^4 = 0, \qquad \omega_0^2 = n_*^2 + x_1^2$$
$$\omega_1 = \omega_0^{-1} \{(\sin 2x_1 - 2x_1)^{-1} [{}^1/_3 x_1^3 + {}^1/_3 n_*^2 x_1 - n_*^4(2b_0 x_1^{-3} + {}^2/_3 x_1^{-1})] + n_*^2 [(2 - 2\varkappa)x_1^{-2} - {}^1/_{12}] + n_*^4 [(-{}^{17}/_8 + {}^5/_2\varkappa)x_1^{-4} + {}^1/_{24}x_1^{-2}]\} \tag{4.11}$$

In the case of very large k_{ni} and n_m ($n_m \gg \varepsilon^{-1}$) the roots of (1.20) should be sought from the asymptotic equation (2.15), in which λ_{ki} should be replaced by in_m, and k by ik_{ni}. Thus tranformed, it defines eight roots which satisfy the asymptotic equalities

$$k_{ni} \approx \pm n_m e^{\varepsilon}, \quad k_{ni} \approx \pm n_m e^{-\varepsilon} \quad \text{for} \quad n_m \to \infty \tag{4.12}$$

and the principal parts of the remaining roots are found from the equations

$$\theta = 2k_{ni}\varepsilon i\,(-1 + {}^1/_2\gamma_*^2 \operatorname{sh}2\varepsilon / 2\varepsilon + {}^1/_8\gamma_*^4 \operatorname{sh}4\varepsilon / 4\varepsilon + \ldots) = in\pi$$
$$Q^{**} = \gamma_*^4[k_{ni}^2 \operatorname{sh}^2 2\varepsilon(1 + \gamma_*^2 \operatorname{ch}2\varepsilon + \ldots) + \operatorname{sh}^2\theta(1 + 2\gamma_*^2 \operatorname{ch}2\varepsilon + \ldots)] = 0$$
$$(n = 1,2,\ldots,\gamma_* = n_m / k_{ni}) \tag{4.13}$$

which are obtained from (2.15) by using expansions which are valid when $|\gamma_* e^{\varepsilon}| \leqslant 1$.

Thus the analysis expounded above shows that the second characteristic equation (1.20) contains three groups of roots. In the first group are the quadruple roots $k_{0i} = 0$ and the double roots $k_{0i} = \pm i$ defined for $n_m = 0$. The second group consists of eight roots defined by (4.4), (4.6), (4.7), (4.9), (4.12). The order of the absolute values of these roots hence depends essentially on the quantity n_m. For small n_m ($n_m \lesssim \varepsilon^{1/2}$) the absolute values of four of them are commensurate with $\varepsilon^{-1/2}n_m$ (small roots), and the other four roots are subject to the relationship $|k_{ni}| \approx 1$ (large roots). For medium values of n_m ($n_m \approx 1$) all eight roots are commensurate with $\varepsilon^{-1/4}$ in absolute value. In the case of large and very large n_m ($n_m \approx \varepsilon^{-1}$ and $n_m \gg \varepsilon^{-1}$), the relationships $k_{ni} \approx \pm n_m$ and $k_{ni} \approx \pm n_m \exp(\pm\varepsilon)$, are satisfied, respectively. The third group includes a countable set of roots defined by (4.2), (4.10), (4.11), (4.13) which grow as $1/\varepsilon$ as $\varepsilon \to 0$.

5. Analysis of the state of stress and strain of an open cylindrical shell. Group (1). The stress function

$$\Psi_2^*(\zeta,\varphi) = E_{-1}\varphi + E_0\varphi^2 + E_1\varphi^3 + (K_{1,2} + K_{1,2}^*\varphi)e^{i\varphi} \tag{5.1}$$

corresponds to the quadruple roots $k_{0i} = 0$ and the double roots $k_{0i} = \pm i$, defined for $n_m = 0$, where $E_{-1}, E_0, \ldots, K_{1,2}$ are arbitrary constants.

Utilizing (5.1), (1.6), (1.1 – 3), we find the displacements and stresses

$$u = 0, \qquad v = 0, \quad w = R_1\varphi E_0$$
$$\tau_{z\varphi} = G\rho_1^{-1}E_0, \qquad \sigma_r = \sigma_\varphi = \sigma_z = \tau_{rz} = \tau_{r\varphi} = 0$$
$$u = R_1 [(a_3\rho_1 + \rho_1^{-1})\, c_1^* + (2a_3\ln\rho_1 - \varkappa)\rho_1]E_1, \quad v = R_1 2\varkappa\rho_1\varphi E_1, \quad w = 0 \tag{5.2}$$
$$\sigma_\varphi = 2G\,[c_1^*(1 + \rho_1^{-2}) + 2(\ln\rho_1 + 1)]E_1, \quad \sigma_r = 2G\,[c_1^*(1 - \rho_1^{-2}) + 2\ln\rho_1]E_1$$
$$\sigma_z = -2Ga_2(1 + c_1^* + 2\ln\rho_1)E_1, \quad \tau_{r\varphi} = \tau_{rz} = \tau_{z\varphi} = 0 \tag{5.3}$$
$$u = R_1\,[2a_3(d_1\ln\rho_1 + d_2\rho_1^2) - d_2\rho_1^2 - d_1 + d_3\rho_1^{-2} - i2\varkappa d_1\varphi]K_{1,2}^* e^{i\varphi}$$
$$v = R_1[2a_3(d_1\ln\rho_1 - d_2\rho_1^2) - 3d_2\rho_1^2 + d_1 - d_3\rho_1^{-2} - i2\varkappa d_1\varphi]iK_{1,2}^* e^{i\varphi}, \quad w = 0 \tag{5.4}$$

$$\sigma_\varphi = 4G(3d_2\rho_1 + d_1\rho_1^{-1} + d_3\rho_1^{-3})K_{1,2}{}^*e^{i\varphi}, \qquad \tau_{rz} = 0$$
$$\sigma_z = -4Ga_2(d_1\rho_1^{-1} + 2d_2\rho_1)K_{1,2}{}^*e^{i\varphi}, \qquad \tau_{z\varphi} = 0$$
$$\tau_{r\varphi} = -i\sigma_r = 4G(d_3\rho_1^{-3} - d_1\rho_1^{-1} - d_2\rho_1)iK_{1,2}{}^*e^{i\varphi} \tag{5.5}$$
$$(c_1{}^* = d_0d_3{}^{-1}\ln d_0, \qquad d_3 = 1 - d_0,\ d_2 = 1 - d_0{}^{-1},\ d_1 = d_0{}^{-1} - d_0)$$

It is easy to show that the stress functions (5.1) correspond to the following elementary states of stress: (1) pure shear (E_0), (2) pure bending by edge moments (E_1), (3) bending from the joint effect of a moment and tensile forces applied to the boundary Γ_2 ($K_{1,2}{}^*$).

The constants E_{-1} and $K_{1,2}$ correspond to shell motion as a rigid body.

Group (2). The solutions (3.6), (3.7) in which k should be replaced by ik_{ni}, and p_0 by in_0, and the stress function Ψ_1 by Ψ_2, correspond to roots defined for small n_m ($n_m \lesssim \varepsilon^{1/2}$) by (4.4). The quantities $u, v, \ldots, \tau_{r\varphi}$ will then satisfy the relationships

$$|\tau_{z\varphi}| \approx n_m{}^3, \quad |\sigma_\varphi|, \quad |\sigma_z| \approx k_{ni}n_m{}^2, \quad |\tau_{r\varphi}| \approx \varepsilon k_{ni}{}^2n_m{}^2, \quad |\sigma_r| \approx \varepsilon k_{ni}n_m{}^2$$
$$|\tau_{rz}| \approx \varepsilon k_{ni}n_m{}^3, \quad |v| \approx k_{ni}{}^2, \quad |u| \approx k_{ni}{}^3, \quad |w| \approx k_{ni}n_m$$
$$(|k_{ni}| \approx \varepsilon^{-1/2}n_m\ |k_{ni}| \approx 1) \tag{5.6}$$

Therefore, for $n_m \lesssim \varepsilon^{1/2}$ the state of stress of an open shell is determined by σ_φ and σ_z in a first approximation, and as is seen from (3.7), is primarily bending. Thus, solutions corresponding to the eight roots of the second group for $n_m \lesssim \varepsilon^{1/2}$ are generalized edge effects [8], which decrease as $\exp(-\varepsilon^{-1/2}n_m\gamma^* s_2)$, where $\mathrm{Re}\gamma^* > 0$ and s_2 is the angular distance from the boundary Γ_2.

For medium values of n_m ($\varepsilon^{1/2} < n_m < \varepsilon^{-1/2}$), the roots k_{ni} are determined from (4.6) and their corresponding solutions are given by formulas (*)

$$u = R_3\varepsilon^{-1/2}\{-\varkappa k_2{}^2 - \varepsilon^{1/2}b_{-2}n_m{}^2 + \ldots\}\,\partial\Psi_2/\partial\varphi$$
$$v = R_3\varepsilon^{-1/2}\{\varkappa k_2{}^2 + \varepsilon^{1/2}(\varkappa t k_2{}^4 + 2b_0n_m{}^2) + \ldots\}\,\Psi_2$$
$$w = R_3\{-\varkappa + \varepsilon^{1/2}\varkappa t k_2{}^2 + \ldots\}\,\partial^2\Psi_2/\partial\zeta\partial\varphi$$
$$\sigma_r = 2G\{\varepsilon(t^2 - 1)k_2{}^4 + \ldots\}\,\partial\Psi_2/\partial\varphi \tag{5.7}$$
$$\tau_{r\varphi} = 2G\{\varepsilon^{1/2}(1 - t^2)k_2{}^6 + \ldots\}\,\Psi_2$$
$$\tau_{rz} = 2G\{\varepsilon(1 - t^2)k_2{}^4 + \ldots\}\,\partial^2\Psi_2/\partial\zeta\partial\varphi$$
$$\sigma_z = 2G\{[-b_0n_m{}^2 - a_2tk_2{}^4] + \varepsilon^{1/2}k_2{}^2[(2a_0 - b_0)n_m{}^2t - a_2t - {}^1/_3a_2k_2{}^4] + \ldots\}\,\partial\Psi_2/\partial\varphi$$
$$\sigma_\varphi = 2G\{[2tk_2{}^4] + \varepsilon^{1/2}k_2{}^2[2t(1 + a_0n_m{}^2) + {}^2/_3k_2{}^4] + \ldots\}\,\partial\Psi_2/\partial\varphi$$
$$\tau_{z\varphi} = 2G\{[\varkappa tk_2{}^4 + b_0n_m{}^2] + \varepsilon^{1/2}k_2{}^2[t(\varkappa + b_{-2}n_m{}^2) + {}^1/_3a_2k_2{}^4] + \ldots\}\,\partial\Psi_2/\partial\zeta \tag{5.8}$$

The relationships

$$|u| \approx \varepsilon^{-3/4}n_m{}^{3/2}, \quad |v| \approx \varepsilon^{-1/2}n_m, \quad |w| \approx \varepsilon^{-1/4}n_m{}^{3/2}, \quad |\sigma_z|, \quad |\sigma_\varphi| \approx \varepsilon^{-1/4}n_m{}^{5/2}$$
$$|\tau_{z\varphi}| \approx n_m{}^3, \quad |\tau_{r\varphi}| \approx \varepsilon^{1/2}n_m{}^3, \quad |\tau_{rz}| \approx \varepsilon^{3/4}n_m{}^{7/2}, \quad |\sigma_r| \approx \varepsilon^{3/4}n_m{}^{5/2} \tag{5.9}$$

result from (5.7), (5.8).

Being primarily a bending state, the state of stress (5.8) hence damps out as $\exp(-\varepsilon^{-1/4}n_m{}^{1/2}\gamma^{**}s_2)$ ($\mathrm{Re}\gamma^{**} > 0$). Therefore, the homogeneous solutions corresponding to medium values of n_m ($\varepsilon^{1/2} < n_m < \varepsilon^{-1/2}$), are edge effects whose damping zones

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*) The solutions (5.7), (5.8) can be refined by terms in ε and $\varepsilon^{1/4}$, if the relationships (3.9), (3.10), as well as (3.1), (3.9) from [1], are utilized.

will be the narrower, the greater the $\varepsilon^{-1/4} n_m^{1/2}$.

For large values of n_m ($\varepsilon^{-1/2} \lesssim n_m < \varepsilon^{-1}$ and $n_m \approx \varepsilon^{-1}$), the solutions (3.9), (3.10) and (3.14), (3.15) correspond, respectively, to the roots defined by (4.7) and (4.9), where k should be replaced by ik_{ni}, λ_{ki} by in_m, and Ψ_1 by Ψ_2. The estimates (3.13) are retained for the quantities $u, v, \ldots, \tau_{z\varphi}$ even this time. Hence, all the characteristics of the homogeneous solutions (3.9), (3.10) and (3.14), (3.15) decrease as $\exp(-n_m s_2)$, including the components of both the bending and the membrane states of stress.

Group (3). Presented below are exact solutions of the third group for $n_m = 0$.

$$u = 0, \quad v = 0, \quad w = R_1 \cos\eta \Psi_2, \quad \sigma_z = \sigma_\varphi = \sigma_r = 0$$

$$\tau_{rz} = -G\rho_1^{-1} \sin\eta \Psi_2', \quad \tau_{z\varphi} = G\rho_1^{-1} \cos\eta \Psi_2', \quad \tau_{r\varphi} = 0 \tag{5.10}$$

The k_{0i} in (5.10) are roots of the equation $\sin 2\varepsilon k_{0i} / 2\varepsilon k_{0i} = 0$

$$u = R_1\rho_1 [(2\varkappa - 1) C_\eta' - E_\eta' + \rho_1^{-2} k_{0i} K_\eta'] \Psi_2, \qquad w = 0$$
$$v = R_1\rho_1 [(1 - 2\varkappa) C_\eta - E_\eta + \rho_1^{-2} k_{0i} K_\eta] \Psi_2' / k_{0i}, \quad \tau_{rz} = \tau_{z\varphi} = 0 \tag{5.11}$$

$$\tau_{r\varphi} = 2G(-E_\eta' + \rho_1^{-2} H_\eta') \Psi_2', \qquad \sigma_\varphi = 2G(2E_\eta' - k_{0i} E_\eta + \rho_1^{-2} k_{0i} H_\eta) \Psi_2$$
$$\sigma_r = 2G(2E_\eta' + k_{0i} E_\eta - \rho_1^{-2} k_{0i} H_\eta) \Psi_2, \qquad \sigma_z = -4Ga_2 E_\eta' \Psi_2 \tag{5.12}$$

$$C_\eta = -(\cos\eta + k_{0i} \sin\eta) \pm R_2 / R_1 [\cos(\eta - \theta_1) + k_{0i} \sin(\eta - \theta_1)]$$
$$K_\eta = -(R_2 / R_1)^2 (\sin\eta + k_{0i} \cos\eta) \pm R_2 / R_1 [\sin(\eta - \theta_1) + k_{0i} \cos(\eta - \theta_1)]$$

$$E_\eta = C_\eta + k_{0i} C_\eta', \quad H_\eta = K_\eta' + k_{0i} K_\eta, \quad \eta = k_{0i} \ln \rho_1, \quad \theta_1 = 2\varepsilon k_{0i} \tag{5.13}$$

The primes here denote differentiation with respect to η and φ, and k_{0i} are the non-zero roots of the equations $\sin 2\varepsilon k_{0i} \pm k_{0i} \operatorname{sh} 2\varepsilon = 0$. In the case when $0 < n_m \lesssim \varepsilon^{-1}$, the homogeneous solutions are given in a first approximation by (3.17), (3.18) and (3.20), (3.21) in which the quantities k, λ_{ki} and Ψ_1 are replaced by ik_{ni}, in_m, Ψ_2, respectively, and therefore, the behavior of solutions of the third group of a cylindrical panel are the same as the analogous solutions of a hollow cylinder.

6. Construction of refined applied theories for circular cylindrical shells. As is seen from (3.1)–(3.9) in [1], the homogeneous solutions (1.1)–(1.3) can be represented in three forms

$$\left.\begin{aligned}(u, \sigma_\varphi, \sigma_r, \sigma_z) &= (\Omega_{11}, \Omega_{21}, \Omega_{31}, \Omega_{41}) p\Psi, \quad \tau_{z\varphi} = \Omega_{51} \partial_2 \Psi \\ (v, \tau_{r\varphi}) &= (\Omega_{61}, \Omega_{71}) p\partial_2 \Psi, \quad (w, \tau_{rz}) = (\Omega_{81}, \Omega_{91}) \Psi\end{aligned}\right\} \tag{6.1}$$

$$\left.\begin{aligned}(u, \sigma_\varphi, \sigma_r, \sigma_z) &= (\Omega_{12}, \Omega_{22}, \Omega_{32}, \Omega_{42}) \partial_2 \Psi, \quad \tau_{z\varphi} = \Omega_{52} p\Psi \\ (v, \tau_{r\varphi}) &= (\Omega_{62}, \Omega_{72}) \Psi, \quad (w, \tau_{rz}) = (\Omega_{82}, \Omega_{92}) \partial_2 p\Psi\end{aligned}\right\} \tag{6.2}$$

$$\left.\begin{aligned}(u, \sigma_\varphi, \sigma_r, \sigma_z) &= (\Omega_{13}, \Omega_{23}, \Omega_{33}, \Omega_{43}), \Psi, \quad \tau_{z\varphi} = \Omega_{53} \partial_2 p\Psi \\ (v, \tau_{r\varphi}) &= (\Omega_{63}, \Omega_{73}) \partial_2 \Psi, \quad (w, \tau_{rz}) = (\Omega_{83} \Omega_{93}) p\Psi\end{aligned}\right\} \tag{6.3}$$

Here $(u, \sigma_\varphi, \ldots) = (\Omega_{13}, \Omega_{23}, \ldots)\Psi$ denotes the system of equalities $u = \Omega_{13}\Psi$, $\sigma_\varphi = \Omega_{23}\Psi, \ldots$, and the quantities $\Omega_{j\mu}$ are integer operator functions of D^2, p^2, ε and $\ln\rho$, representable by series of the following form:

$$\Omega_{j\mu} = \sum_{k=0}^{\infty} \varepsilon^k \Omega_{j\mu k}(D^2, p^2, t), \qquad \Omega_{j\mu} = \sum_{k=0}^{\infty} \varepsilon^k \Omega^*_{j\mu k}(D_*^2, p_*^2, t_1) \tag{6.4}$$

where the operators $\Omega_{j\mu k}$ and $\Omega^*_{j\mu k}$ are of the type (1.12) and (1.17).

Therefore, if the stress function Ψ has an index of variability $p^\circ \lesssim \varepsilon^{-1}$ ($p^\circ = \max\{k, n_m, |k_{ni}|, |\lambda_{ki}|\}$), then by keeping a sufficient number of terms in the series (6.1)–(6.4), a series of applied theories of circular cylindrical shells can be constructed which have any previously assigned accuracy in ε. Hence, by having the solutions of the third group, the boundary conditions can be satisfied more accurately than in the integral sense [9, 10]. In this case a system of algebraic equations in H_{ki} and H_{ni}^* is obtained, which separates asymptotically, for small ε, into one eighth order system and two countable infinite order systems (see e. g. [3]). These latter have been studied in [6, 11], and are solved effectively by the method of reduction.

As regards the constraint imposed on the index of variability p°, it is insignificant since such theories are intended to reduce smoothly varying external loadings applied to the boundary Γ_2. The relationships herein are given as a specific refined applied theory. Together with the relationships (3.1), (3.9) from [1], the proposed theory yields an error on the order of ε^2 as compared with unity, if $p^\circ \lesssim \varepsilon^{-1/2}$, and an error on the order of ε if $p^\circ \approx \varepsilon^{-1}$, and can be utilized to check the accuracy of existing applied theories.

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